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Elastic Stability of Annular Plates under Uniform Compressive Forces along the Outer Edge

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I. Introduction

SEVERAL investigations on the problem of elastic stability of annular plates subjected to uniform radial pressure along the outer edge have been reported in the literature.[1–5] Most of these investigations are confined to plates with a free inner edge and clamped or simply supported outer edge. In the present Note, the title problem is analyzed in detail for all nine combinations of clamped, simply supported, and free-edge conditions. The analysis is carried out by the classical Rayleigh-Ritz method with simple polynomials as admissible functions. For annuli of narrow width, suitable coordinate transformations proposed earlier[6,7] are incorporated so as to improve the conditioning of the equations.

II. Analysis

A thin annular plate of constant thickness h, with a and b as the radii of inner and outer edges, respectively, and subjected to uniform inplane radial pressure p_o along the outer edge is considered. Assuming that the plate buckles in n circumferential waves, the lateral deflection $W(r, \theta)$ is expressed as

$$W(r,\theta) = W_n(r)\cos(n\theta+\varepsilon) \tag{1}$$

The strain energy V of bending and the potential T due to midplane forces during bending of the plate are then given by

$$V = \frac{\pi}{2}(1+\delta_{on})D\int_a^b\left[\left(\frac{d^2W_n}{dr^2}+\frac{1}{r}\frac{dW_n}{dr}-\frac{n^2}{r^2}W_n\right)^2 - 2(1-\nu)\frac{d^2W_n}{dr^2}\left(\frac{1}{r}\frac{dW_n}{dr}-\frac{n^2}{r^2}W_n\right) + 2(1-\nu)\frac{n^2}{r^2}\left(\frac{dW_n}{dr}-\frac{W_n}{r}\right)^2\right]r\,dr \tag{2}$$

and

$$T = \frac{\pi}{2}(1+\delta_{on})h\left[\int_a^b\left\{\sigma_r\left(\frac{dW_n}{dr}\right)^2+\sigma_\theta\left(\frac{n}{r}W_n\right)^2\right\}r\,dr\right] \tag{3}$$

in which $\delta_{oo} = 1$ and $\delta_{on} = 0$ for $n \neq 0$, ν is the Poisson's ratio, $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity of the plate, and

$$\sigma_r = -\frac{p_o b^2}{b^2-a^2}\left(1-\frac{a^2}{r^2}\right) \tag{4}$$

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Table 1 Transformations

n	First transformation	Second transformation
0	$W_o(r) = w(r)$	$y = (r^2-a^2)/(b^2-a^2)$
1	$W_1(r) = rw(r)$	$y = (r-a)/(b-a)$
$\geqq 2$	$W_n(r) = r^2 w(r)$	$y = (r-a)/(b-a)$

$$\sigma_\theta = -\frac{p_o b^2}{b^2-a^2}\left(1+\frac{a^2}{r^2}\right) \tag{5}$$

are the prebuckling membrane stresses.

Following the usual steps in the application of the Rayleigh-Ritz method, the total potential $V+T$ is minimized to obtain relevant characteristic equations for determination of buckling loads. Admissible functions used in the method are chosen to be simple polynomials in r.[6] This procedure, however, is known[6,7] to lead to an ill-conditioned set of equations for annuli of narrow width, particularly in the case of both edges clamped. To overcome this difficulty in computational work, the quantities V and T are expressed, as in earlier analyses,[6,7] in more convenient forms by means of the transformations in Table 1. After obtaining these new expressions for V and T, the Rayleigh-Ritz method is applied with simple polynomials in y as admissible functions.[6] Unlike in the direct analysis, the conditioning of the equations derived thus is found to improve with increasing hole size.

III. Numerical Results and Conclusions

Extensive data for buckling loads are obtained by using direct analysis for $a/b \leqq 0.3$ and by modified analysis for $a/b > 0.3$. In all calculations, the Poisson's ratio $\nu = 0.3$ is used. In view of

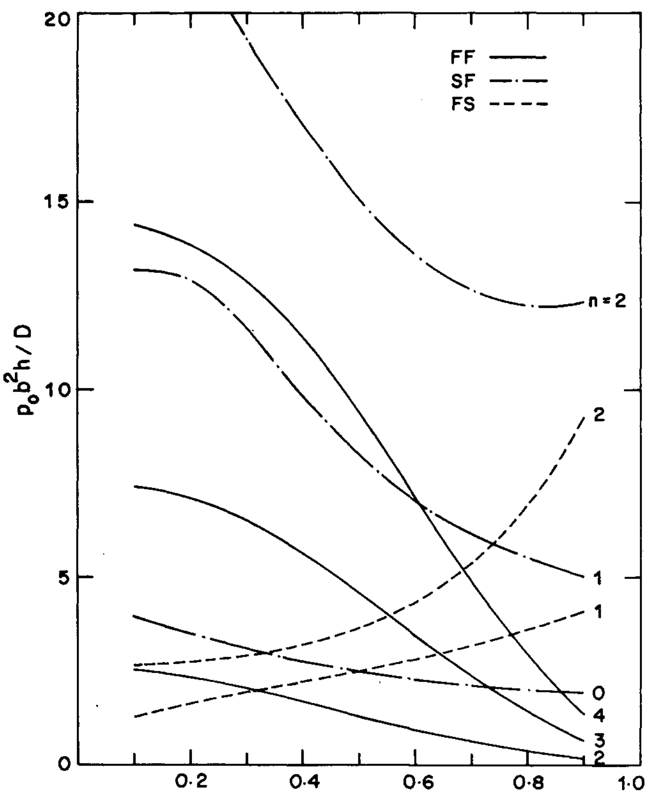

Fig. 1 Variation of buckling load parameter with hole size; ——— both edges free (FF), —·— outer edge simply supported and inner edge free (SF), - - - - outer edge free and inner edge simply supported (FS). The buckling loads for $n = 0$ are identical in these three cases.[8]

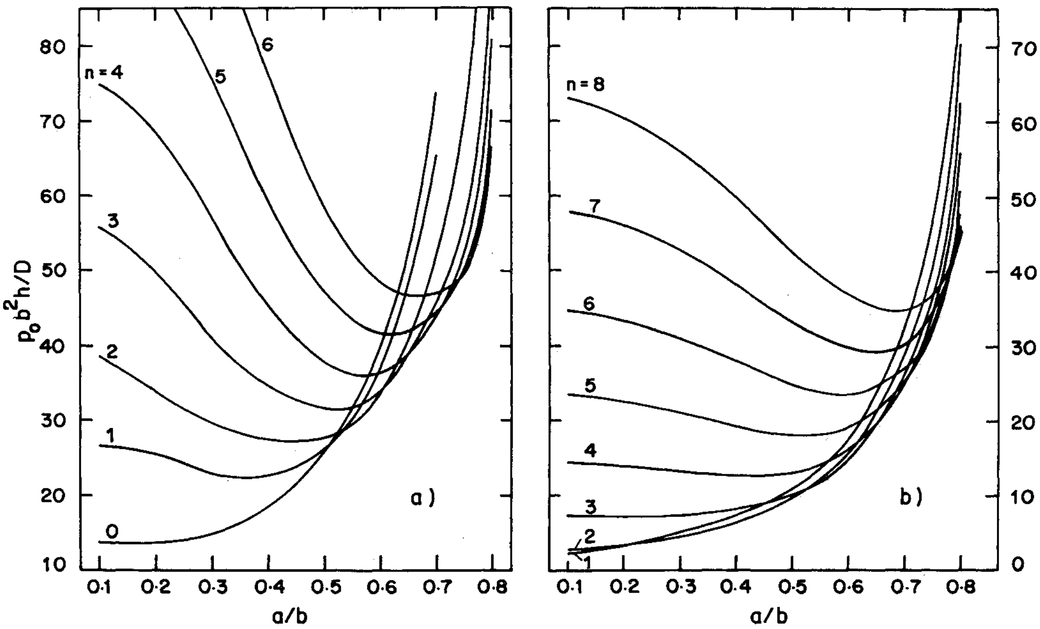

Fig. 2 Variation of buckling load parameter with hole size: a) outer edge clamped and inner edge free (CF), b) outer edge free and inner edge clamped (FC).

the earlier experience with regard to convergence and accuracy of solutions by present analyses, the numerical results obtained are believed to be quite accurate. The variation of buckling load parameter $(p_o b^2 h/D)$ with hole size a/b is shown in Figs. 1–4 for modes with various number of nodal diameters and for various edge conditions.

In the three cases of one edge free and the other either simply supported or free, the axisymmetric buckling loads are known[8] to be identical. Hence, the curve for $n = 0$ in the SF case in Fig. 1 also represents the corresponding curves in the FS and FF cases. It is to be noted that the axisymmetric buckling load in the FF case corresponds to the mode with one nodal circle. In fact, this load corresponds to the first axisymmetric mode of a free plate supported along any concentric circle.[6]

The data in Fig. 1 show that the plate buckles in two circumferential waves in the FF case and axisymmetrically in the SF case for all hole sizes. In the FS case, the plate buckles with one circumferential wave for hole sizes less than about 0.5 and axisymmetrically for hole sizes greater than 0.5. From these observations, one may infer that a free plate with circumferential ring support mentioned earlier may buckle with one circumferential wave if the hole size is less than 0.5 and the location of the ring support is nearer to the inner edge; otherwise, the free plate will buckle axisymmetrically. It is also interesting to note from Fig. 1 that, for hole sizes less than about 0.3, the critical buckling load of the free plate is higher than that of the plate supported along the inner edge. The data presented for higher modes in the three cases of SF, FS, and FF correspond to the

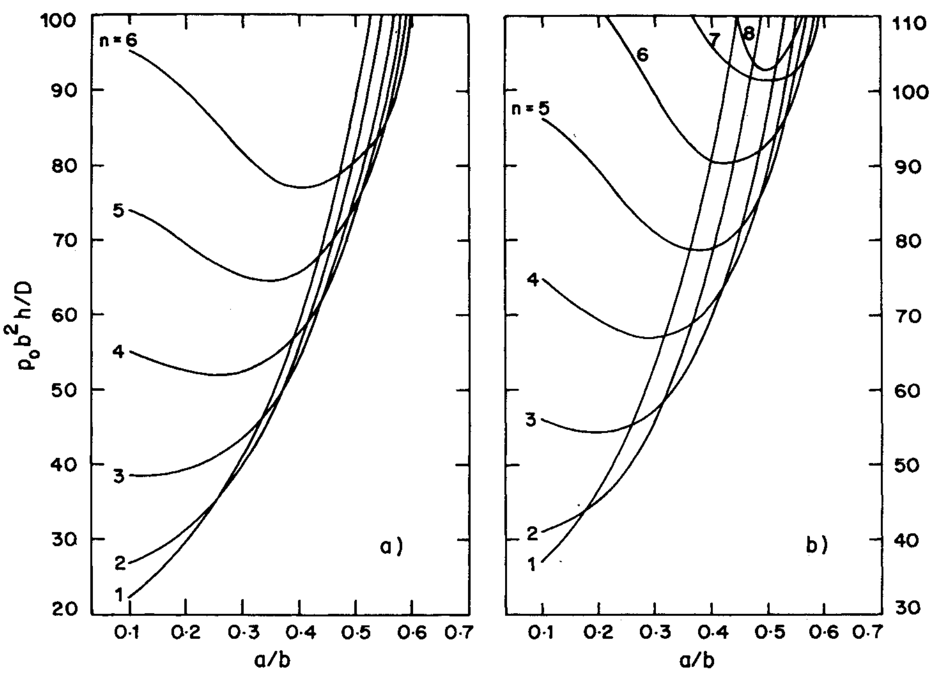

Fig. 3 Variation of buckling load parameter with hole size: a) outer edge simply supported and inner edge clamped (SC), b) outer edge clamped and inner edge simply supported (CS).

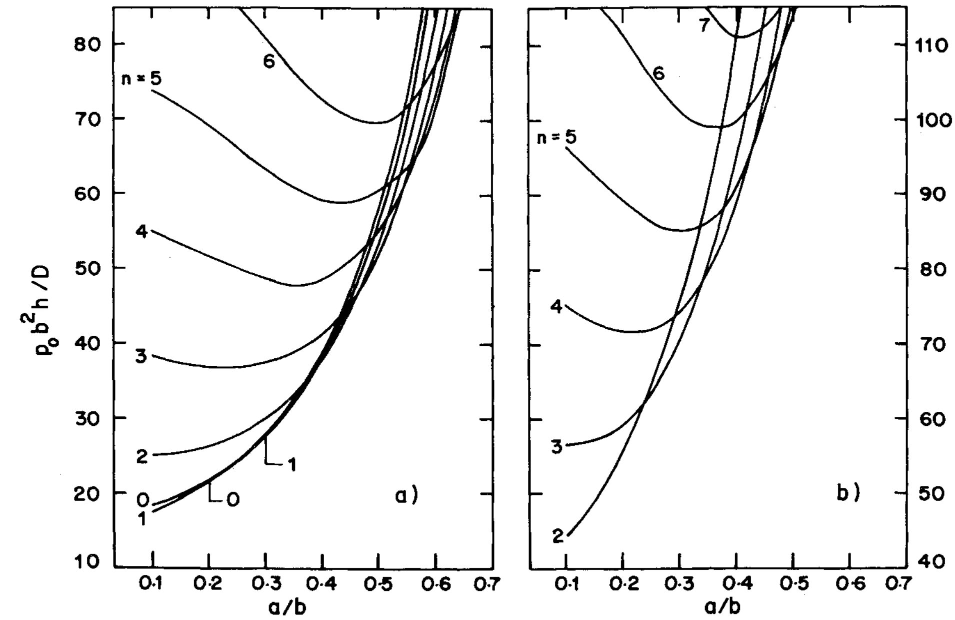

Fig. 4 Variation of buckling load parameter with hole size: a) both edges simply supported (SS); b) both edges clamped (CC).

critical buckling loads of plates supported along appropriate number of regularly spaced diametral lines.

Out of the remaining six cases (Figs. 2–4), the plate buckles axisymmetrically in the CF case for hole sizes up to about 0.5 (Fig. 2a) and in the SS case for hole sizes between 0.2–0.3 (Fig. 4a). For other hole sizes in the CF and SS cases and for all hole sizes in the remaining four cases, the plate buckles with one or more circumferential waves. In such cases, the first axisymmetric eigenmode and the corresponding eigenload have no physical significance whatsoever. However, it may be mentioned here that the higher axisymmetric modes do correspond to the critical buckling modes of plates supported along the relevant nodal circles. The modes with higher number of circumferential waves considered in the present investigation have similar physical significance, as now explained. Let $(a/b)_n$ denote the hole size of the plate for which the two modes with n and $2n$ number of circumferential waves yield equal buckling loads. Then for a/b less than $(a/b)_n$, the mode with n circumferential waves is the critical buckling mode of the plate supported along n regularly spaced diametral lines. For a/b greater than $(a/b)_n$, this mode and the corresponding eigenvalue have no physical significance.

From Figs. 2–4 it can also be seen that the number of circumferential waves of the buckled surface increases, in general, with increasing hole size and with increasing geometric constraints at the edges. For these critical buckling modes of plates with large holes, Majumdar[4] proposed one term Rayleigh-Ritz procedure in the CF case and stated that such a procedure "provides a reasonable estimate of the minimum buckling loads for at least up to values of $a/b = 0.9$ and $n = 10$." However, such estimates have been found to be more than 5 percent higher than the present accurate estimates.

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Numerical Methods for Evaluating the Derivatives of Eigenvalues and Eigenvectors

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I. Introduction

MATHEMATICAL programing methods for optimizing the design of systems where the dynamic stability and/or response of the system is a function of several design parameters often require derivatives of the eigenvalues and eigenvectors of a characteristic equation of the system. Cwach and Stearman[1] utilized the first derivatives of the eigenvalues of the flutter equation of a lifting surface to find gradients of the damping factor of the flutter characteristic equation. These gradients were used to optimize an active control system which suppressed the flutter of a lifting surface. Rudisill and Bhatia[2,3] utilized

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